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# AN OPTIMUM DESIGN OF ELEMENTS INTRODUCING A LOAD INTO A SPHERICAL SHELL

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**Abstract**—The paper deals with the design of a shape of an element introducing the tangential load into the spherical shell in a way avoiding strong concentration of stresses in the shell. The shape of the attachment is obtained by solving two equations of compatibility of normal and tangential displacements of the shell and the attachment.

## 1. INTRODUCTION

The problem of introducing concentrated loads into thin-walled structures is very important from the technical point of view. These structures are usually designed taking into account certain general loads which result from the basic nature of the structure. For example, the thickness of the shell of a container for gas or liquid is designed taking into account the internal pressure. The stresses resulting from the local loads or from discontinuities in the structure are often disregarded. But we know, that the structure can be damaged as a result of an improper introduction of relatively small local loads. This problem is especially important in the case of fatigue loads when the local concentration of stresses is most dangerous. The subject of this paper is the design of a shape of the element introducing the load in the spherical shell in the form of advantageously distributed tangential and normal forces. Under the term "advantageously distributed forces" we understand the forces which do not produce a strong concentration of stresses in the shell. It is known that infinitely large stresses can appear not only in the case when the load acts on a very small area of the structure but also when the load is introduced in an improper way. Let us consider a following example. We introduce the load into a sheet through a bar of constant cross-section jointed with the sheet along a certain segment. Solving this problem by means of the equations of the linear theory of elasticity we find that the infinitely large stresses appear in the sheet at the ends of the segment of the joint. Even in the case when we apply the load to the sheet in the form of the line forces uniformly distributed along the segment of the joint the singular stresses appear at the beginning and at the end of this segment. We can, however, avoid the concentration of stresses by appropriate design of the shape of the bar, by changing for example its cross-section. The paper [1] was devoted to the analysis of this problem.

## 2. CONDITIONS OF COMPATIBILITY

Let us consider two bodies jointed over the surface $S$. The following equilibrium and compatibility conditions should be satisfied (Fig. 1)

$$T_1 = T_2 = T$$

$$u_1 = u_2$$

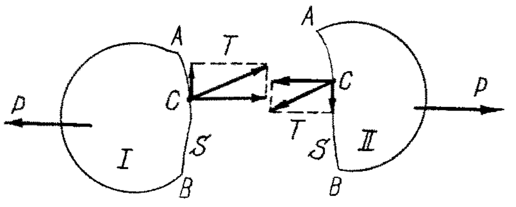



Fig. 1.

where $T_i$ are the vectors of the surface forces in the junction,
$u_i$ are the vectors of the displacements.

Indices 1 and 2 denote the first and second body respectively. The compatibility equations can be presented in the form of the integral equations

$$\int_S K_1 T \, ds = \int_S K_2 \, T \, ds \tag{1}$$

where $K_1$ and $K_2$ are the influence functions (Green's functions) for the displacements or strains for the first and second body. $S$ is the surface of the junction. If we know the form of both bodies and their boundary conditions, we obtain, solving the equation (1), the distribution of the forces acting between these bodies. These equations can be used, however, for designing an appropriate shape of the jointed bodies if we consider the Green's functions as the unknowns and assume the advantageous distribution of the forces in the junction. If we look only for certain parameters of the influence functions, the equation (1) becomes ordinary algebraic equations which can be solved easily.

Let us consider particularly the case of the spherical shell loaded by a force $P$ acting in the direction tangential to the shell surface.

We assume on the surface of the shell the local system of nondimensional coordinates $x = \bar{x}/l$, $y = \bar{y}/l$ referred to the characteristic length $l = \sqrt{Rh}/\sqrt[4]{12(1 - V^2)}$ with the origin at the equator. The direction of the coordinate $x$ follows the direction of the meridian and that of the coordinate $y$ the direction of the equator.

Let us introduce the force into the spherical shell through a thin flat attachment jointed with the shell along the line $x$ and perpendicular to the shell surface (Fig. 3).

The shape of the attachment is the aim of our investigations. We shall define it in such a way that the forces of the reciprocal actions between the shell and the attachment do not produce strong concentration of stresses in the shell.

Since the attachment is directly connected with the shell, the following compatibility conditions should be satisfied in the common points of the shell and the edge of the attachment

$$v_S = v_R \qquad w_S = w_R$$

where $v_S$, $v_R$ and $w_S$, $w_R$ are the tangential and normal displacements of the shell and the attachment respectively. Instead of these conditions the following more convenient conditions can be applied,

$$\varepsilon_S = \varepsilon_R, \qquad w_S'' = w_R''$$

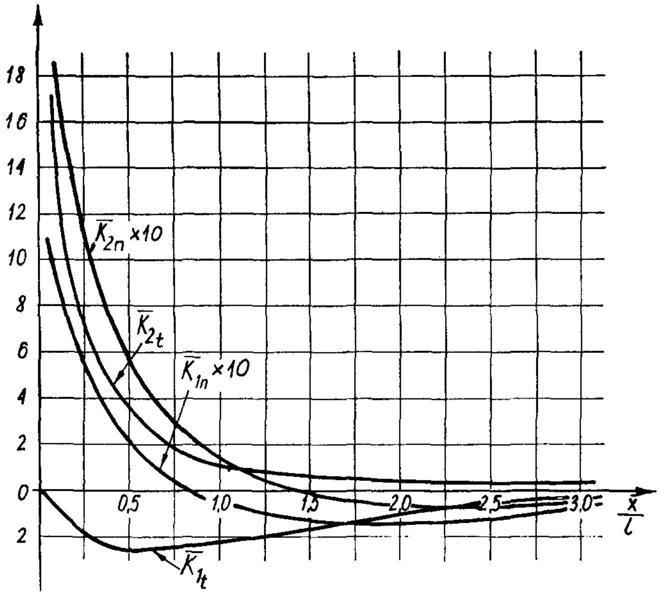



Fig. 2.

where $\varepsilon_S$ and $\varepsilon_R$ are the strains along the junction and $w_S''$, $w_R''$ are the changes of the curvature; $w'' = \partial^2 w/\partial x^2$. If we assume that between the shell and the attachment act the normal and tangential line loads only we can present the above conditions in the form of the following integral equations.

$$w_S'' = -\int_{-a}^{+a} [\overline{K}_{1n}(x, \xi)n(\xi) + \overline{K}_{1t}(x, \xi)q(\xi)]\mathrm{d}\xi = w_R''$$
$$\varepsilon_S = \int_{-a}^{a} [\overline{K}_{2n}(x, \xi)n(\xi) + \overline{K}_{2t}(x, \xi)q(\xi)]\mathrm{d}\xi = \varepsilon_R \qquad (2)$$

where $x = \bar{x}/l = \varphi R/l$, $\xi = \bar{\xi}/l = \psi R/l$, $a = \varphi_0 R/l$, are the non-dimensional coordinates referred to the characteristic length $l$; $q(\xi)$ and $n(\xi)$ are the tangential and normal reactions between the shell and the attachment. The quantities $\overline{K}_{1n}$, $\overline{K}_{2n}$, $\overline{K}_{1t}$ and $\overline{K}_{2t}$ are the influence functions for the spherical shell.

$\overline{K}_{1n}(x, \xi)$ denotes the change of the curvature $\partial^2 w/\partial x^2$ produced at the point $x$ by the normal unit load $n = 1$ applied to the shell at the point $\xi$.

$\overline{K}_{1t}(x, \xi)$ denotes the change of the curvature $\partial^2 w/\partial x^2$ produced at the point $x$ by the unit tangential load $q = 1$ applied at the point $\xi$.

$\overline{K}_{2n}(x, \xi)$ denotes the strain $\varepsilon_{xx}$ produced at the point $x$ by the normal unit load $n = 1$ applied at the point $\xi$.

$\overline{K}_{2t}(x, \xi)$ denotes the strain $\varepsilon_{xx}$ produced at the point $x$ by the tangential unit load $q = 1$ and applied at the point $\xi$.

These functions can be found on the basis of the solution for the spherical shell loaded by concentrated unit tangential and normal forces [2]. We have

$$\bar{K}_{1n} = w''(n = 1) = -\frac{1}{2\pi D}\left[ker(x-\xi) - \frac{1}{(x-\xi)}kei'(x-\xi)\right]$$

$$\bar{K}_{1t} = w''(q = 1) = (1+v)\frac{kl}{2\pi RD}\left[\frac{2}{(x-\xi)^2}\left(\frac{1}{(x-\xi)} + ker'(x-\xi)\right)\right. \tag{3}$$

$$\left. + \frac{1}{x-\xi}kei(x-\xi) - kei'(x-\xi)\right].$$

The strains in the external surface of the shell are defined by following equation:

$$\varepsilon_{xx} = \frac{1}{E_s h_s}(N_{rr} - vN_\theta) + \frac{6}{E_s h_s^2}(M_{rr} - vM_\theta); \qquad y = 0$$

then the functions $\bar{K}_{2n}$ and $\bar{K}_{2t}$ are [2]

$$\bar{K}_{2n} = \varepsilon_{xx}(n = 1) = \frac{\sqrt{3(1-v^2)}}{E_s h_s^2}\left[\frac{1+v}{x-\xi}\left(ker'(x-\xi) + \frac{1}{x-\xi}\right)\right.$$

$$\left. + vkei(x-\xi) + \sqrt{3(1-V^2)}\left(ker(x-\xi) - \frac{1}{x-\xi}kei'(x-\xi)\right)\right].$$

$$\bar{K}_{2t} = \varepsilon_{xx}(q = 1) = -\frac{(1+v)k}{2\pi E_s h_s l}\left\{\left[1 - (1+v)\left(ker(x-\xi) - \frac{2}{x-\xi}kei'(x-\xi)\right)\right]\frac{1}{x-\xi}\right. \tag{4}$$

$$+ vker'(x-\xi) + \sqrt{3(1-v^2)}\left[\frac{2}{(x-\xi)^2}\left(\frac{1}{x-\xi} + ker'(x-\xi)\right)\right.$$

$$\left.\left. + \frac{1}{x-\xi}kei(x-\xi) - kei'(x-\xi)\right]\right\}.$$

where $k = 1$ for $x > \xi$ and $k = -1$ for $x < \xi$. The introduction of the nondimensional functions is convenient for numerical calculations. We denote

$$K_{1n} = -\frac{1}{2\pi D}\bar{K}_{1n}$$

$$K_{1t} = \frac{1+v}{2}\frac{l}{DR}\bar{K}_{1t} \tag{5}$$

$$K_{2n} = \frac{\sqrt{3(1-v^2)}}{E_s h_s^2}\bar{K}_{2n}$$

$$K_{2t} = \frac{1}{2\pi E_s h_s}\bar{K}_{2t}; \qquad D = \frac{E_s h_s^3}{12(1-v^2)}.$$

The above functions are presented graphically in Fig. 3 for $v = 0{\cdot}3$.

## 3. DISPLACEMENTS AND STRAINS OF THE ATTACHMENT

If we consider the attachment as a flat sheet, the calculation of the strains at its edge is rather complicated. Assuming that the attachment is a slender element, we can calculate its

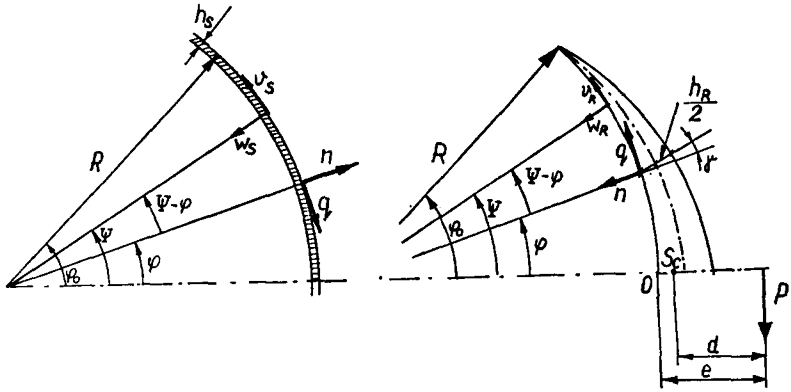



Fig. 3.

strains and displacements in the same way as for a bar under bending and tension. The strains of the attachment in the points where it is jointed with the shell are

$$\varepsilon_{xR} = \frac{M}{W_R E_R} + \frac{N}{A_R E_R} \tag{6}$$

where $A_R$ is the surface of the cross-section of the attachment, $W_R$ is the section modulus of the attachment, and $E_R$ is the modulus of elasticity of the material of the attachment. The internal force $N$ and the bending moment $M$ can be calculated from the equilibrium equations in the following form

$$\begin{aligned} N &= \int_{\varphi}^{\varphi_0} [q(\psi)\cos(\psi - \varphi) - n(\psi)\sin(\psi - \varphi)]Rd\psi \\ M &= \int_{\varphi}^{\varphi_0} \left[\left(R + \frac{h_R}{2}\cos\gamma\right)\cos(\psi - \varphi) - R\right] q(\psi)Rd\psi \\ &\quad - \int_{\varphi}^{\varphi_0}\left(R + \frac{h_R}{2}\cos\gamma\right)\sin(\psi - \varphi)n(\psi)Rd\psi. \end{aligned} \tag{7}$$

If the height of the attachment $h_R$ is not too big in comparison with the radius of the shell, it can be assumed that $\cos\gamma \cong 1$ (Fig. 3). Then the strain in the attachment along the line of the joint takes the form

$$\begin{aligned} \varepsilon_{xR} = \frac{1}{A_R E_R}\int_{\varphi}^{\varphi_0}\Bigg\{&\cos(\psi - \varphi) - \frac{RA_R}{W_R}\left[\left(1 + \frac{h_R}{2R}\right)\cos(\psi - \varphi) - 1\right]q(\psi)Rd\psi \\ &- \int_{\varphi}^{\varphi_0}\left[\frac{1}{E_R A_R} + \frac{R}{E_R W_R}\left(1 + \frac{h_R}{2R}\right)\right]\sin(\psi - \varphi)n(\psi)Rd\psi\Bigg\}. \end{aligned} \tag{8}$$

The change of the curvature $w_R''$ can be found from the equation

$$\begin{aligned} w_R'' = \frac{M}{E_R J_R} = \frac{1}{E_R J_R}\Bigg\{&\int_{\varphi}^{\varphi_0} R\left[\left(1 + \frac{h_R}{2R}\right)\cos(\psi - \varphi) - 1\right]q(\psi)Rd\psi \\ &- \int_{\varphi}^{\varphi_0} R\left(1 + \frac{h_R}{2R}\right)\sin(\psi - \varphi)n(\psi)Rd\psi\Bigg\}. \end{aligned} \tag{9}$$

The equations (2) together with (6), (8), (9) define the solution of the given problem. If we are looking only for reaction forces $q$ and $n$, these equations create the set of two integral equations in respect to these two functions. However, if we consider as the unknowns the parameters determining the shape of the attachment, we obtain the set of two algebraic equations which can be solved immediately. Since we have to satisfy two equations we can define only two parameters determining the shape of the attachment.

Forces $n$ and $q$ should be assumed in such a form that they do not produce singular stresses in the shell. This can be done if, for example

$$q(\xi) = q_0\left[1 - \left(\frac{\xi}{a}\right)^2\right]^2, \qquad n(\xi) = n_0\left[1 - \left(\frac{\xi}{a}\right)^2\right]^2 \frac{\xi}{a} \tag{10}$$

where

$$\bar{\xi} = R\psi, \qquad \xi = \bar{\xi}/l = \psi R/l, \qquad \bar{a} = \varphi_0 R, \qquad a = \bar{a}/l.$$

It can be easily proved that the forces distributed in such a way ensure relatively uniform effort of the material of the shell in the area of the joint. The values $q_0$ and $n_0$ can be found from the equilibrium equations of the attachment. Namely, the conditions of the equilibrium of the moments give the relation

$$P(e + R) = \int_{-a}^{a} q(\xi)Rd\xi$$

where $e$ is the distance between the point of the application of the load and the external surface of the shell. The condition of the equilibrium of the attachment may be expressed by

$$\int_{-\varphi_0}^{\varphi_0} [-n(\psi)\sin\psi + q(\psi)\cos\psi]Rd\psi = P.$$

We find from the first equation

$$q_0 = \frac{15}{16}\frac{e + R}{R}\frac{P}{a}.$$

The second equation expresses the relation between the force $n_0$ and the distance $e$.

## 4. ATTACHMENT OF RECTANGULAR CROSS-SECTION

If we assume that the cross-section of the attachment is a rectangle of the dimensions $h_R$ and $b_R$ we have

$$A_R = b_R h_R, \qquad W_R = b_R h_R^2/6, \qquad J_R = b_R h_R^3/12.$$

Introducing it to the equations (1) we obtain the following set of equations

$$\frac{4A(\varphi)}{b_R h_R} - \frac{6RB(\varphi)}{b_R h_R^2} = \varepsilon_{xS}, \qquad \frac{6A(\varphi)}{b_R h_R^2} - \frac{12B(\varphi)}{b_R h_R^3} = w_S'' \tag{11}$$

$$A(\varphi) = \frac{1}{E_R}\int_{\varphi}^{\varphi_0} [\cos(\psi - \varphi)q(\psi) - \sin(\psi - \varphi)n(\psi)]Rd\psi$$
$$B(\varphi) = \frac{1}{E_R}\int_{\varphi}^{\varphi_0} \{[1 - \cos(\psi - \varphi)]q(\psi) + \sin(\psi - \varphi)n(\psi)\}Rd\psi. \tag{12}$$

Solving this set we find the following equation for $h_R$.

$$h_R^2 - \frac{3}{2}\left(\eta R + \frac{\varepsilon_{xS}}{w''}\right) h_R + 3\eta R \frac{\varepsilon_{xS}}{w''} = 0; \qquad \eta = \frac{B}{A}.$$

The solution takes the form:

$$h_{R(1,2)} = \frac{3}{4} R \left[\frac{1}{2}\frac{\tilde{h}_R}{R} + \eta \pm \eta\sqrt{1 - \frac{5}{3}\frac{\tilde{h}_R}{R\eta} + \frac{\tilde{h}_R^2}{4R^2\eta^2}}\right] \tag{13}$$

where

$$\tilde{h}_R = \frac{2\varepsilon_{xS}}{w''}.$$

If $B/A \ll 1$ we obtain from (13)

$$h_R \cong \tilde{h}_R$$

where the sign $-$ corresponds to the real value of the height of the attachment. The real solution for the case when $B/A$ is not a small value exists only if

$$\eta^2 - \frac{5}{3}\frac{\tilde{h}_R}{R}\eta + \frac{\tilde{h}_R^2}{4R^2} \geq 0.$$

From this inequality we have two conditions

$$\frac{\tilde{h}_R}{R\eta} \geq 3 \quad \text{or} \quad \frac{\tilde{h}_R}{R\eta} \leq \frac{1}{3}.$$

The formula for the thickness of the attachment $b_R$ can be found from one of the equations (11). For example, from the first equation we get

$$b_R = \frac{1}{\varepsilon_{xS}}\left[\frac{4A}{h_R} - \frac{6BR}{h_R^2}\right]. \tag{14}$$

The relations (13) and (14) define the shape of the attachment for the assumed distribution of the tangential and normal forces acting between the attachment and the shell.

Figures 4 and 5 present the height and the width of the attachment for the different ratios of $a/l$ and $a/R = 0{\cdot}3$, when the load is applied in the shear center of the tangential forces

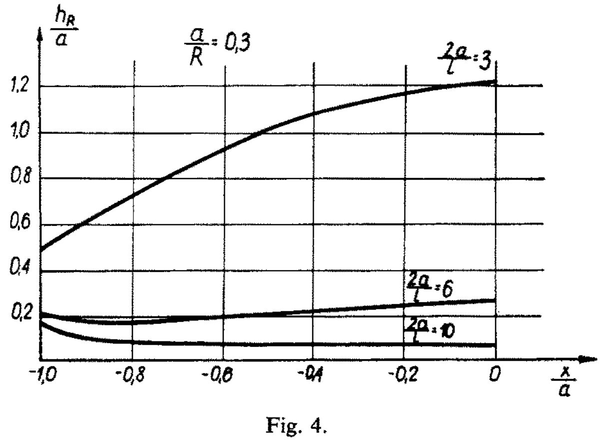



Fig. 4.

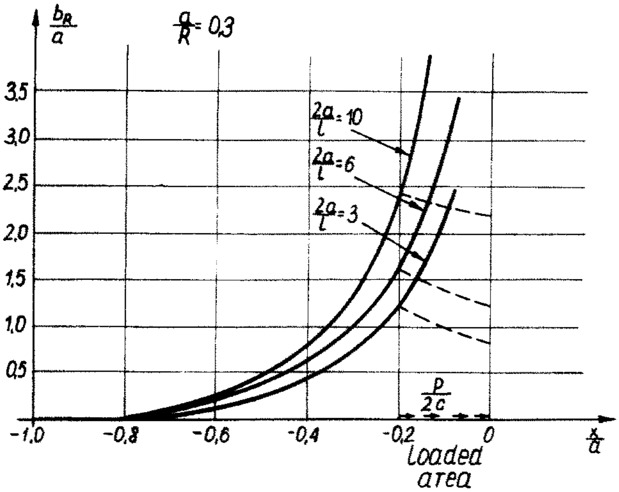



Fig. 5.

$q(\xi)$. It gives $n(\xi) = 0$. If the load is applied only in one cross-section of the attachment we observe that its width increases there to infinity. The finite width is obtained for the case when the load is distributed along a certain distance. The broken lines in Fig. 5 correspond to the case when the force $P$ is uniformly distributed along the distance $2c = 0{\cdot}4a$.

It is interesting that the height of the attachment does not decrease to zero at the ends of the joint. However, the bending and tensile rigidity of the attachment is there equal to zero. For thinner shells the height of the attachment becomes smaller and the attachment has a form of a slice of almost constant thickness. Assuming that the attachment works like a bar, we have to assume at the same time that the shear forces are distributed linearly along the joint. In the real structure the attachment is usually welded to the shell at the boundary and therefore the assumed model of the joint is only an approximated model. The result of the calculations relating to the case when the force $P$ applied to the shell does not lie in the shear center are presented in Figs. 6 and 7. The curves for the height are obtained for the

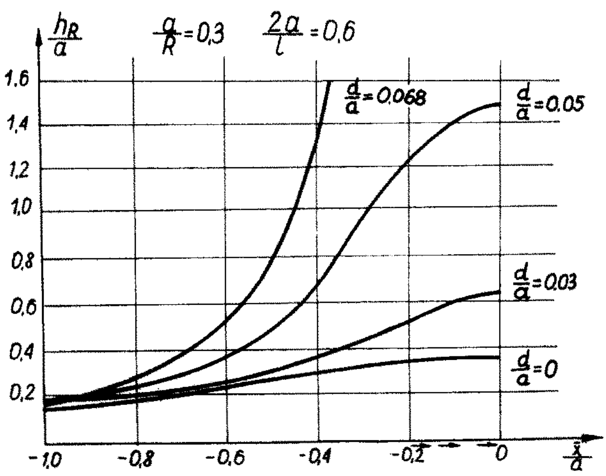
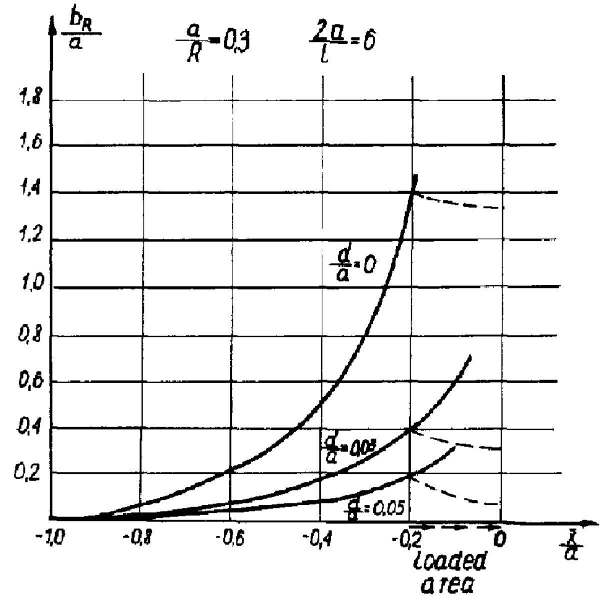
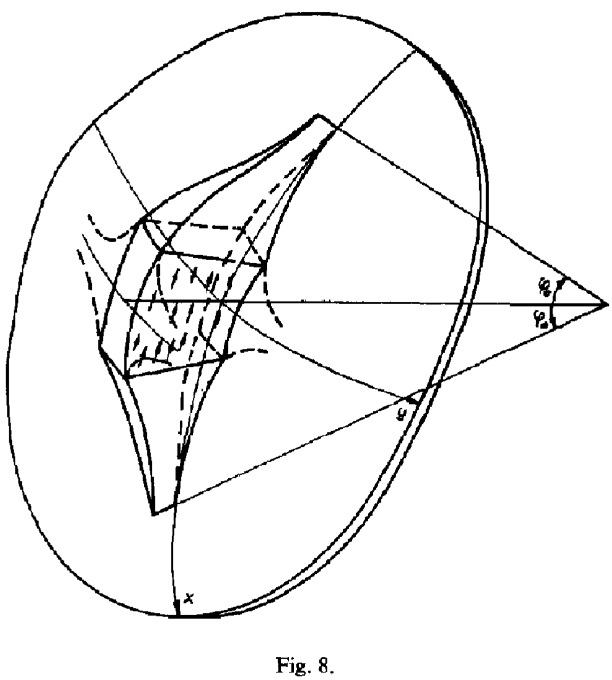



Fig. 6.



$\frac{b_R}{a}$
$\frac{a}{R}=0.3$
$\frac{2a}{l}=6$
1,8
1,6
1,4
1,2
1,0
0,8
0,6
0,4
0,2
$\frac{d}{a}=0$
$\frac{d}{a}=0.05$
-1,0
-0,8
-0,6
-0,4
-0,2
0
$\frac{\bar{x}}{a}$
loaded
area


Fig. 7.



x


Fig. 8.

different values of the ratio $d/a$ (see Fig. 3) and $a/R = 0{\cdot}3$, $2a/l = 6$ $d$ is the distance of the shear center from the point of application of the load. The width of the attachment is shown in Fig. 7. Figure 8 shows the shape of the attachment for the case when the load is uniformly distributed along the distance $2c = 0{\cdot}4a$. This shape results from the Figs. 4 and 5 for value of the ratio $d/a = 0{\cdot}05$. The broken lines present the shape of the attachment when the load is applied in one cross-section.

## REFERENCES

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**Резюме** — Настоящая работа посвящена проектированию формы элемента вводящего нагрузку, направленную по касательной, в сферическую оболочку таким образом, чтобы избежать сильную концентрацию напряжений в оболочке. Форму приспособления получают путем решения двух уравнений совместимости нормального и касательного смеЖений оболочки и приспособления.